A Tale of Two Set Theories

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Abstract. We describe the relationship between two versions of Tarski-Grothendieck set theory: the first-order set theory of Mizar and the higher-order set theory of Egal. We show how certain higher-order terms and propositions in Egal have equivalent first-order presentations. We then prove Tarski's Axiom A (an axiom in Mizar) in Egal and construct a Grothendieck Universe operator (a primitive with axioms in Egal) in Mizar.

Keywords: Formalized Mathematics, Theorem Proving, Set Theory, Proof Checking, Mizar

1 Introduction

We compare two implemented versions of Tarski-Grothendieck (TG) set theory. The first is the first-order TG implemented in Mizar [3,13] axiomatized using Tarski's Axiom A [21,22]. The other is the higher-order TG implemented in Egal [6] axiomatized using Grothendieck universes [15]. We discuss what would be involved in porting Mizar developments into Egal and vice versa.

We use Egal's Grothendieck universes (along with a choice operator) to prove Tarski's Axiom A in Egal. Consequently the Egal counterpart of each of Mizar's axioms is provable in Egal and so porting from Mizar to Egal should always be possible in principle. In practice one would need to make Mizar's implicit reasoning using its type system explicit, a nontrivial task outside the scope of this paper.

Porting from Egal to Mizar poses two challenges. One is that many definitions and propositions in Egal make use of higher-order quantifiers. In order to give a Mizar counterpart, it is enough to give a first-order reformulation and prove the two formulations equivalent in Egal. While this will not always be possible in principle, it has been possible for the examples necessary for this paper. The second challenge is to construct a Grothendieck universe operator in Mizar that satisfies the properties of a corresponding operator in Egal. We have constructed such an operator.

We give a brief introduction to Mizar and its version first-order Tarski-Grothendieck in Section 2 and an introduction to Egal and its version of higher-order Tarski-Grothendieck in Section 3. In Section 4 we give a few examples of definitions and propositions in Egal that can be reformulated in equivalent first-order forms. These first-order versions have counterparts in Mizar. Section 5 discusses the Egal proof of Tarski's Axiom A. In Section 6 we discuss the construction of a Grothendieck universe operator in Mizar.[3]

2 Mizar and FOTG

The Mizar system [14] from its beginning aimed to create a proof style that simultaneously imitates informal mathematical proofs as much as possible and and can be automatically verified to be logically correct. A quite simple and intuitive reasoning formalism and an intuitive soft type system play a major role in the pursuit of Mizar's goals.

The Mizar proof style is mainly inspired by Jaśkowski [16] style of natural deduction and most statements correspond to valid first-order predicate calculus formulas. Over time the Mizar community has also added support for syntax that goes beyond traditional first-order terms and formulas. In particular, Mizar supports **schemes** with predicate and function variables, sufficient to formulate the Fraenkel replacement as one axiom in Mizar. This axiom is sufficient to construct the set comprehension $\{Fx|x \in X, Px\}$ (called *Fraenkel terms*) for a given set X, function F and predicate P in the Mizar language but it is impossible to define such a functor for arbitrary X, F, P. Therefore, in response to the needs of Mizar's users, support for Fraenkel terms has been built into the system. In fact Mizar supports a generalized notation where the set membership relation $x \in X$ in the Fraenkel term has been replaced by the type membership $x : \Theta$ if the Mizar type Θ has the **sethood** property. A Mizar type has the **sethood** property if the collection of all objects of the type forms a set (as opposed to a class). Semantically, Mizar types are simply unary first-order predicates over sets that can be parameterized by sets. However, the type inference mechanisms make Mizar significantly more powerful and user-friendly. The rules available for automatic type inference are influenced by the author of a given script by choosing the **environ** (i.e., environment, see [13]). By skillfully choosing the environment, an author can make a Mizar article more concise and readable since the type system will handle many inferences implicitly. Mizar types must be inhabited and this obligation must be proven by a user directly in the definition of a given type or before the first use if a type has the form of intersection of types.

Parallel to the system development, the Mizar community puts a significant effort into building the Mizar Mathematical Library (MML)[4]. The MML is the comprehensive repository of currently formalized mathematics in the Mizar system. The foundation of the library, up to some details discussed below, is first-order Tarski-Grothendieck set theory (FOTG). This is a non-conservative extension Zermelo–Fraenkel set theory (ZFC), where the axiom of infinity has been replaced by Tarski's Axiom A. The statement of Axiom A in Mizar is shown in Figure 1.

FOTG was not the only foundation considered for the library. One of the main reasons it was chosen is the usefulness of the Axiom A in the formaliza-

[3] At `http://grid01.ciirc.cvut.cz/~chad/twosettheories.tgz` one can find Egal, the Egal formalization files and the Mizar formalization files.

```
reserve N,M,X,Y,Z for set;
theorem :: TARSKI_A:1
  ex M st N in M &
     (for X,Y holds X in M & Y c= X implies Y in M) &
     (for X st X in M ex Z st Z in M & for Y st Y c= X holds Y in Z) &
     (for X holds X c= M implies X,M are_equipotent or X in M);
```

Fig. 1. Tarski's Axiom A in Mizar

tion of category theory. Namely, FOTG provides many universes that that have properties analogous to those that have classes of all sets. In particular, every axiom of ZFC remains true if we relativize quantifiers to the given universe.

The axiom of choice can be proven in FOTG. Bancerek used Axiom A to prove Zermelo's well-ordering theorem and the axiom of choice [2]. Later changes to Mizar also yielded the axiom of choice in a more direct way. While working with category theory in the Mizar system, Trybulec decided to introduce a sophisticated construction called *permissive* definition (implemented in Mizar-2 in the 80's [14]). Permissive definitions allowed an author to introduce an inhabited type morphism of a,b under the assumption that there exists a morphism from a to b. It is important to note that this construction, in contrast to Fraenkel terms, cannot be semantically justified in FOTG since the construction allows the definition of a choice operator for any type Θ of a,b,... in the following way:

```
definition
  let a,b,... such that C: contradiction;
  func choose(a,b,...) → Θ of a,b,... means contradiction;
  existence by C; uniqueness by C;
end;
```

To avoid repetition of such definitions, in 2012, the Mizar syntax was extended by the *explicit* operator **the** (e.g., **the** Θ **of** a,b,...). This new operator behaves similarly to a Hilbert ε-operator, which corresponds to have a global choice operator on the universe of sets (cf. p. 72 of [11]). ZFC extended with a global choice operator is known to be conservative over ZFC [10]. The situation with FOTG is analogous to that of ZFC, and we conjecture FOTG extended with a global choice operator (**the**) is conservative over FOTG. Regardless of the truth of this conjecture, we take the proper foundation of the MML to be FOTG extended with a global choice operator (see [17]).

3 Egal and HOTG

Egal [6] is a proof checker for higher-order Tarski-Grothendieck (HOTG) set theory. The idea of combining higher-order logic and set theory is not new [12,18,20]. Egal differs from previous efforts by restricting the "higher-order logic" to a simple type theory in the style of Church [8] with no complicating additions such as type definitions. In addition, Egal places an emphasis on proof terms and proof checking. Egal proof scripts are presented in a way similar to Coq [5] and instruct Egal how to construct a proof term.

The base of the Egal system includes simply typed λ-calculus with a type of propositions along with a λ-calculus for proof terms. There is a base type of individuals ι (thought of as sets), a based type of propositions o and function types $\sigma \to \tau$. Egal is designed to have its foundational framework simple while still allowing nontrivial formalizations.

Without extra axioms, the logic of Egal is intentional intuitionistic higher-order logic. On top of this logic we add constants and axioms that yield an extensional classical higher-order set theory.

To be precise let $\mathcal{T}$ be the set of types generated freely via the grammar $o|\iota|\sigma \to \tau$. We use σ, τ to range over types. For each $\sigma \in \mathcal{T}$ let $\mathcal{V}_\sigma$ be a countably infinite set of variables and assume $\mathcal{V}_\sigma \cap \mathcal{V}_\tau = \emptyset$ whenever $\sigma \neq \tau$. We use $x, y, z, X, Y, f, g, p, q, P, Q, \ldots$ to range over variables. For each $\sigma \in \mathcal{T}$ let $\mathcal{C}_\sigma$ be a set of constants. We use c, c_1, c_2 to range over constants. We consider only a fixed family of constants given as follows:

- ε_σ is a constant in $\mathcal{C}_{(\sigma\to o)\to\sigma}$ for each type σ.
- In is a constant in $\mathcal{C}_{\iota\to\iota\to o}$.
- Empty is a constant in $\mathcal{C}_\iota$.
- Union is a constant in $\mathcal{C}_{\iota\to\iota}$.
- Power is a constant in $\mathcal{C}_{\iota\to\iota}$.
- Repl is a constant in $\mathcal{C}_{\iota\to(\iota\to\iota)\to\iota}$.
- UnivOf is a constant in $\mathcal{C}_{\iota\to\iota}$.

No other constants are allowed. We assume none of these constants are variables.

We next define a family $(\Lambda_\sigma)_{\sigma\in\mathcal{T}}$ of typed terms as follows. We use s, t and u to range over terms.

- If $x \in \mathcal{V}_\sigma$, then $x \in \Lambda_\sigma$.
- If $c \in \mathcal{C}_\sigma$, then $c \in \Lambda_\sigma$.
- If $s \in \Lambda_{\sigma\to\tau}$ and $t \in \Lambda_\sigma$, then $(st) \in \Lambda_\tau$.
- If $x \in \mathcal{V}_\sigma$ and $t \in \Lambda_\tau$, then $(\lambda x.t) \in \Lambda_{\sigma\to\tau}$.
- If $s \in \Lambda_o$ and $t \in \Lambda_o$, then $(s \Rightarrow t) \in \Lambda_o$.
- If $x \in \mathcal{V}_\sigma$ and $t \in \Lambda_o$, then $(\forall x.t) \in \Lambda_o$.

Each member of Λ_σ is a *term of type* σ. Terms of type o are also called *propositions*. We sometimes use φ, ψ and ξ to range over propositions. It is easy to see that Λ_σ and Λ_τ are disjoint for $\sigma \neq \tau$. That is, each term has at most one type.

We omit parentheses when possible, with application associating to the left and implication associating to the right: stu means $((st)u)$ and $\varphi \Rightarrow \psi \Rightarrow \xi$ means $(\varphi \Rightarrow (\psi \Rightarrow \xi))$. Binders are often combined: $\lambda xyz.s$ means $\lambda x.\lambda y.\lambda z.s$ and $\forall xyz.\varphi$ means $\forall x.\forall y.\forall z.\varphi$. To present the types of variables concisely, we often annotate variables in binders with their types, as in $\lambda x : \sigma.s$ to assert $x \in \mathcal{V}_\sigma$. When the type of a variable is omitted entirely, it is ι.

Although the only logical connectives as part of the definition of terms are implication and universal quantification, it is well-known how to define the other connectives and quantifiers in a way that even works in an intuitionistic setting [7]. For this reason we freely write propositions $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \Leftrightarrow \psi)$, $(\exists x.\varphi)$ and $(s = t)$ (for $s, t \in \Lambda_\sigma$). Again, we omit parentheses and use common binder abbreviations in obvious ways.

We also use special notations for terms built using the constants. We write $s \in t$ for $\mathsf{In}\ s\ t$. We write $\forall x \in s.\varphi$ for $\forall x.x \in s \Rightarrow \varphi$ and $\exists x \in s.\varphi$ for $\exists x.x \in s \wedge \varphi$. We write $\varepsilon x : \sigma.\varphi$ for $\varepsilon_\sigma(\lambda x : \sigma.\varphi)$ and $\varepsilon x \in s.\varphi$ for $\varepsilon x.x \in s \wedge \varphi$. We also write $\emptyset$ for Empty, $\bigcup s$ for $\mathsf{Union}\ s$, $\wp s$ for $\mathsf{Power}\ s$, $\{s|x \in t\}$ for $\mathsf{Repl}\ s\ (\lambda x.t)$ and $\mathcal{U}_N$ for $\mathsf{UnivOf}\ s$.

In general new names can be introduced to abbreviate terms of a given type. In many cases we introduce new corresponding notations as well. The following abbreviations are used in the statements of the axioms below:

- $\mathsf{TransSet} : \iota \to o$ is $\lambda U.\forall X \in U.X \subseteq U$. Informally we say U *is transitive* to mean $\mathsf{TransSet}\ U$.
- $\mathsf{Union\_closed} : \iota \to o$ is $\lambda U.\forall X \in U.\bigcup X \in U$. Informally we say U *is* $\bigcup$*-closed* to mean $\mathsf{Union\_closed}\ U$.
- $\mathsf{Power\_closed} : \iota \to o$ is $\lambda U.\forall X \in U.\wp X \in U$. Informally we say U *is* $\wp$*-closed* to mean $\mathsf{Power\_closed}\ U$.
- $\mathsf{Repl\_closed} : \iota \to o$ is $\lambda U.\forall X \in U.\forall F : \iota \to \iota.(\forall x \in X.Fx \in U) \Rightarrow \{Fx|x \in X\} \in U$. Informally we say U *is closed under replacement* to mean $\mathsf{Repl\_closed}\ U$.
- $\mathsf{ZF\_closed} : \iota \to o$ is $\lambda U.\mathsf{Union\_closed}\ U \wedge \mathsf{Power\_closed}\ U \wedge \mathsf{Repl\_closed}\ U$. Informally we say U *is ZF-closed* to mean $\mathsf{ZF\_closed}\ U$.

The deduction system for Egal includes a set $\mathcal{A}$ of closed propositions we call axioms. The specific members of the set $\mathcal{A}$ are as follows:

Prop. Ext. $\forall PQ : o.(P \Leftrightarrow Q) \Rightarrow P = Q$,
Func. Ext. $\forall fg : \sigma \to \tau.(\forall x : \sigma.fx = gx) \Rightarrow f = g$ (for types σ and τ),
Choice $\forall p : \sigma \to o.\forall x : \sigma.px \Rightarrow p(\varepsilon x : \sigma.px)$ (for each type σ),
Set Ext. $\forall XY.X \subseteq Y \Rightarrow Y \subseteq X \Rightarrow X = Y$,
$\in$-Induction $\forall P : \iota \to o.(\forall X.(\forall x \in X.Px) \Rightarrow PX) \Rightarrow \forall X.PX$,
Empty $\neg\exists x.x \in \emptyset$,
Union $\forall Xx.x \in \bigcup X \Leftrightarrow \exists Y.x \in Y \wedge Y \in X$,
Power $\forall XY.Y \in \wp X \Leftrightarrow Y \subseteq X$,
Replacement $\forall X.\forall F : \iota \to \iota.\forall y.y \in \{Fx|x \in X\} \Leftrightarrow \exists x \in X.y = Fx$,
Universe In $\forall N.N \in \mathcal{U}_N$,
Universe Transitive $\forall N.\mathsf{TransSet}\ \mathcal{U}_N$,
Universe ZF closed $\forall N.\mathsf{ZFclosed}\ \mathcal{U}_N$ and
Universe Min $\forall NU.N \in U \Rightarrow \mathsf{TransSet}\ U \Rightarrow \mathsf{ZFclosed}\ U \Rightarrow \mathcal{U}_N \subseteq U$.

The axiom set would be finite if it were not for functional extensionality and choice. There are well-known ways to formulate functional extensionality and choice as rules in the deduction calculus that would allow them to be removed as axioms while keeping the same set of theorems. We take them as axioms here to keep the deduction system faithful to the implementation in Egal.

The notions of free and bound variables are defined as usual, as is the notion of a variable x being free in a term s. We consider terms equal up to bound variables names. As usual there are notions of capture-avoiding substitution and we write s^x_t to be the result of subsituting t for x in s. We have the usual notions of β-conversion and η-conversion: $(\lambda x.s)t$ β-reduces to s^x_t and $(\lambda x.sx)$ η-reduces to s if x is not free in s. The relation $s \sim_{\beta\eta} t$ on terms $s, t \in \Lambda_\sigma$ is the least congruence relation closed under β-conversion and η-conversion.

The underlying deduction system for Egal is natural deduction with proof terms. We do not discuss proof terms here, but give the corresponding natural deduction calculus without proof terms in Figure 2. The calculus defines when $\Gamma \vdash \varphi$ is derivable where Γ is a finite set of propositions and φ is a proposition.

$$\textsc{Ax}\ \frac{\varphi \in \mathcal{A}}{\Gamma \vdash \varphi} \qquad \textsc{Hyp}\ \frac{\varphi \in \Gamma}{\Gamma \vdash \varphi} \qquad \beta\ \frac{\Gamma \vdash \psi \qquad \psi \sim_{\beta\eta} \varphi}{\Gamma \vdash \varphi} \qquad \Rightarrow\!\mathrm{I}\ \frac{\Gamma \cup \{\varphi\} \vdash \psi}{\Gamma \vdash \varphi \Rightarrow \psi}$$

$$\Rightarrow\!\mathrm{E}\ \frac{\Gamma \vdash \varphi \Rightarrow \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \qquad \forall\mathrm{I}\ \frac{\Gamma \vdash \varphi^x_y \qquad y \in \mathcal{V}_\sigma \text{ is not free in } \Gamma \cup \{\varphi\}}{\Gamma \vdash \forall x : \sigma.\varphi}$$

$$\forall\mathrm{E}\ \frac{\Gamma \vdash \forall x : \sigma.\varphi \qquad t \in \Lambda_\sigma}{\Gamma \vdash \varphi^x_t}$$

Fig. 2. Natural deduction sytem

In addition to the constants and axioms of the system, we import a number of constructions and results from the library distributed with Egal. Some of the constructions are definitions of logical connectives, equality and existential quantification as well as basic theorems about their properties. Negation of equality, negation of set membership and subset are imported, defined in the obvious ways. We use the notation $s \neq t$, $s \notin t$ and $s \subseteq t$ for the corresponding propositions. The definitions $\mathsf{TransSet}$, $\mathsf{Union\_closed}$, $\mathsf{Power\_closed}$, $\mathsf{Repl\_closed}$ and $\mathsf{ZF\_closed}$ are imported. In addition the following definitions are imported:

- $\mathsf{ordinal} : \iota \to o$ is $\lambda\alpha.\mathsf{TransSet}\ \alpha \wedge \forall \beta \in \alpha.\mathsf{TransSet}\ \beta$. Informally we say β *is an ordinal* to mean $\mathsf{ordinal}\ \beta$.
- $\mathsf{famunion} : \iota \to (\iota \to \iota) \to \iota$ is $\lambda XF.\bigcup\{Fx|x \in X\}$. We write $\bigcup_{x\in s} t$ for $\mathsf{famunion}\ s\ (\lambda x.t)$.

We also import the following objects in an opaque way, so that we will only be able to use properties imported from the library and not the actual definitions.

- $\mathsf{Sep} : \iota \to (\iota \to o) \to \iota$. We write $\{x \in X|\varphi\}$ for $\mathsf{Sep}\ X\ (\lambda x.\varphi)$. Results are imported to ensure $\forall z.z \in \{x \in X|\varphi\} \Leftrightarrow z \in X \wedge \varphi^x_z$ is provable.
- $\mathsf{ReplSep} : \iota \to (\iota \to o) \to (\iota \to \iota) \to \iota$. We write $\{s|x \in X \text{ such that } \varphi\}$ for $\mathsf{ReplSep}\ X\ (\lambda x.\varphi)\ (\lambda x.s)$. Results are imported to ensure the provability of $\forall z.z \in \{s|x \in X \text{ such that } \varphi\} \Leftrightarrow \exists y \in X.\varphi^x_y \wedge z = s^x_y$.

– $\mathsf{UPair} : \iota \to \iota \to \iota$. We write $\{x, y\}$ for $\mathsf{UPair}\ x\ y$. Results are imported to ensure $\forall z.z \in \{x, y\} \Leftrightarrow z = x \lor z = y$ is provable.
– $\mathsf{Sing} : \iota \to \iota$. We write $\{x\}$ for $\mathsf{Sing}\ x$. Results are imported to ensure $\forall z.z \in \{x\} \Leftrightarrow z = x$ is provable.
– $\mathsf{R} : (\iota \to (\iota \to \iota) \to \iota) \to \iota \to \iota$. The R operator is used to define functions by $\in$-recursion over the universe. Its construction is discussed in [7] but here we will only need the fundamental property imported as Proposition 5 below. Its use will be essential in proving Tarski's Axiom A in Section 5.

We will freely make use of these imported terms to form new terms below.

Less than 60 results proven in the library need to be imported in order to prove the results discussed in this paper. Most of those results are basic results about logic and set theory and we will leave them implicit here. The choice axiom and the extensionality axioms make the logic extensional and classical [9]. We import excluded middle and the double negation law from the library.

The following imported results are worth making explicit:

Proposition 1. $\forall x.x \notin x$.

Proposition 2 (Regularity). $\forall Xx.x \in X \Rightarrow \exists Y \in X.\neg\exists z \in X.z \in Y$.

Proposition 3. $\forall \alpha.\mathsf{ordinal}\ \alpha \Rightarrow \forall \beta \in \alpha.\mathsf{ordinal}\ \beta$.

Proposition 4. $\forall \alpha\beta.\mathsf{ordinal}\ \alpha \Rightarrow \mathsf{ordinal}\ \beta \Rightarrow \alpha \in \beta \lor \alpha = \beta \lor \beta \in \alpha$.

The fundamental property of R is imported from the library:

Proposition 5 (cf. Theorem 1 in [7]).

$$\forall \Phi : \iota \to (\iota \to \iota) \to \iota.(\forall X.\forall gh : \iota \to \iota.(\forall x \in X.gx = hx) \Rightarrow \Phi\ X\ g = \Phi\ X\ h) \\ \to \forall X.\mathsf{R}\ \Phi\ X = \Phi\ X\ (\mathsf{R}\ \Phi)$$

4 Higher-order vs. First-order Representations

Tarski's Axiom A (Figure 1) informally states that every set is in a Tarski universe. The most interesting condition in the definition of a Tarski universe is that every subset of the universe is either a member of the universe or is equipotent with the universe. The notion of equipotence of two sets can be represented in different ways. In first-order one can define when sets X and Y are equipotent as follows: there is a set R of Kuratowski pairs which essentially encodes the graph of a bijection from X to Y. In order to state Axiom A in Mizar, one must first define Kuratowski pairs and then equipotence. This first-order definition of equipotence can of course be made in Egal as well. We omit the details, except to say we easily obtain an Egal abbreviation equip of type $\iota \to \iota \to o$ with a definition analogous to the definition of equipotence in Mizar.

There is an alternative way to characterize equipotence in Egal without relying on the set theoretic encoding of pairs and functions. We simply use functions of type $\iota \to \iota$ given by the underlying simple type theory.

Let $\mathsf{bij} : \iota \to \iota \to (\iota \to \iota) \to o$ be

$$\lambda XY.\lambda f : \iota \to \iota.(\forall u \in X.fu \in Y) \wedge (\forall uv \in X.fu = fv \Rightarrow u = v)$$
$$\wedge(\forall w \in Y.\exists u \in X.fu = w).$$

Informally we say f *is a bijection taking* X *onto* Y to mean $\mathsf{bij}\ X\ Y\ f$.

It is straightforward to prove $\mathsf{equip}\ X\ Y \Leftrightarrow \exists f : \iota \to \iota.\mathsf{bij}\ X\ Y\ f$ in Egal. When proving Axiom A in Egal (see Theorem 5) we will use $\exists f : \iota \to \iota.\mathsf{bij}\ X\ Y\ f$ to represent equipotence. To obtain the first-order formulation Axiom A, the equivalence of the two formulations of equipotence can be used.

A similar issue arises when considering the notion of being ZF-closed in Mizar. The definition of ZF_closed relies on Repl_closed. Repl_closed relies on the higher-order Repl operator and quantifies over the type $\iota \to \iota$. An alternative first-order definition of U being ZF-closed is to say U is $\wp$-closed and U is closed under internal family unions. The internal family union of a set I and a set f is defined as the set famunionintern $I\ f$ such that $w \in$ famunionintern $I\ f$ if and only if $\exists i \in I.\exists X.[i, X] \in f \wedge w \in X$ where $[i, X]$ is the Kuratowski pair $\{\{i\}, \{i, X\}\}$. It is easy to prove such a set exists, in both Egal and Mizar. Closure of U under internal family unions states that if $I \in U$, f is a set of Kuratowski pairs representing the graph of a function from I into U, then famunionintern $I\ f \in U$.

We say U is *ZF-closed in the FO sense* if U is $\wp$-closed and closed under internal family unions. In Egal it is straightforward to prove that for transitive sets U, U is ZF-closed if and only if U is ZF-closed in the FO sense. Grothendieck universes in Egal are transitive ZF-closed sets. Grothendieck universes in Mizar are transitive sets that are ZF-closed in the FO sense. By the equivalence result, we know these two notions of Grothendieck universes are equivalent in Egal.

5 Tarski's Axiom A in Egal

We will now describe the HOTG proof of Tarski's Axiom A in Egal.

We begin by using the recursion operator to define an operator returning the set of all sets up to a given rank.

– $\mathsf{V} : \iota \to \iota$ is $\mathsf{R}(\lambda Xv.\bigcup_{x\in X} \wp(vx))$. We will write $\mathbf{V}_X$ for V applied to X.

Using Proposition 5 it is easy to prove the following:

Theorem 1. $\forall X.\mathbf{V}_X = \bigcup_{x\in X} \cdot\wp(\mathbf{V}_x)$

It is then straightforward to prove a sequence of results.

Theorem 2. *The following facts hold.*

1. $\forall yxX.x \in X \Rightarrow y \subseteq \mathbf{V}_x \Rightarrow y \in \mathbf{V}_X$.
2. $\forall yX.y \in \mathbf{V}_X \Rightarrow \exists x \in X.y \subseteq \mathbf{V}_x$.
3. $\forall X.X \subseteq \mathbf{V}_X$.
4. $\forall XY.X \subseteq \mathbf{V}_Y \Rightarrow \mathbf{V}_X \subseteq \mathbf{V}_Y$.
5. $\forall XY.X \in \mathbf{V}_Y \Rightarrow \mathbf{V}_X \in \mathbf{V}_Y$.
6. $\forall XY.X \in \mathbf{V}_Y \vee \mathbf{V}_Y \subseteq \mathbf{V}_X$.

7. $\forall XY.\mathbf{V}_X \in \mathbf{V}_Y \vee \mathbf{V}_Y \subseteq \mathbf{V}_X$.

Proof. Parts 1 and 2 are easy consequences of Theorem 1 and properties of powersets and family unions. Part 3 follows by $\in$-induction using Part 1. Part 4 also follows by $\in$-induction using Parts 1 and 2. Part 5 follows easily from Parts 1, 2 and 4. Part 6 follows by $\in$-induction using classical reasoning and Parts 1 and 2. Part 7 follows from Part 5 and 6.

Let $\mathsf{V\_closed}$ of type $\iota \to o$ be $\lambda U.\forall X \in U.\mathbf{V}_X \in U$. Informally we say U if $\mathbf{V}$-closed to mean $\mathsf{V\_closed}\ U$. The following theorem is easy to prove by $\in$-induction using Theorem 1.

Theorem 3. *If U is transitive and ZF-closed, then U is $\mathbf{V}$-closed.*

Using the choice operator it is straightforward to construct the inverse of a bijection taking X onto Y and obtain a bijection taking Y onto X.

Theorem 4. $\forall XY.\forall f : \iota \to \iota.\mathsf{bij}\ X\ Y\ f \Rightarrow \mathsf{bij}\ Y\ X\ (\lambda y.\varepsilon x \in X.fx = y)$.

We now turn to the most complicated Egal proof. More than half of the file ending with the proof of Axiom A is made up of the proof of Lemma 1. For this reason we describe the proof in some detail (though informally) and make some comments about the corresponding formal proof in Egal along the way.

Lemma 1. *Let U be a ZF-closed transitive set and X be such that $X \subseteq U$ and $X \notin U$. There is a bijection $f : \iota \to \iota$ taking $\{\alpha \in U|\mathsf{ordinal}\ \alpha\}$ onto X.*

Proof. In the Egal proof we begin by introducing the local names U and X and making the corresponding assumptions.

```
let U. assume HT: TransSet U. assume HZ: ZF_closed U.
let X. assume HXsU: X c= U. assume HXniU: X /:e U.
```

We next make six local abbreviations. Let

- $\boldsymbol{\lambda}$ be $\{\alpha \in U|\mathsf{ordinal}\ \alpha\}$,
- $\mathbf{P} : \iota \to \iota \to (\iota \to \iota) \to o$ be $\lambda\alpha x f.x \in X \wedge \forall\beta \in \alpha.f\beta \neq x$,
- $\mathbf{Q} : \iota \to (\iota \to \iota) \to \iota \to o$ be $\lambda\alpha f x.\mathbf{P}\ \alpha\ x\ f \wedge \forall y.\mathbf{P}\ \alpha\ y\ f \Rightarrow \mathbf{V}_x \subseteq \mathbf{V}_y$,
- $\mathbf{F} : \iota \to (\iota \to \iota) \to \iota$ be $\lambda\alpha f.\varepsilon x.\mathbf{Q}\ \alpha f x$,
- $\mathbf{f} : \iota \to \iota$ be $\mathsf{R}\mathbf{F}$ and
- $\mathbf{g} : \iota \to \iota$ be $\lambda y.\varepsilon\alpha \in \boldsymbol{\lambda}.\mathbf{f}\alpha = y$.

In the Egal proof three of these local definitions are given as follows:

```
set lambda : set := {alpha :e U|ordinal alpha}.
...
set f : set->set := In_rec F.
set g : set->set := fun y => some alpha :e lambda, f alpha = y.
```

We will prove the following claims:

$$\forall \alpha. \mathbf{f}\alpha = \mathbf{F}\ \alpha\ \mathbf{f} \quad (1)$$
$$\forall \alpha \in \boldsymbol{\lambda}. \mathbf{Q}\ \alpha\ \mathbf{f}\ (\mathbf{f}\alpha) \quad (2)$$
$$\forall \alpha \in \boldsymbol{\lambda}. \mathbf{f}\alpha \in X \quad (3)$$
$$\forall \alpha\beta \in \boldsymbol{\lambda}. \mathbf{f}\alpha = \mathbf{f}\beta \Rightarrow \alpha = \beta \quad (4)$$
$$\mathsf{bij}\ \{\mathbf{f}\ \alpha | \alpha \in \boldsymbol{\lambda}\}\ \boldsymbol{\lambda}\ \mathbf{g} \quad (5)$$
$$\boldsymbol{\lambda} = \{\mathbf{g}\ y | y \in \{\mathbf{f}\ \alpha | \alpha \in \boldsymbol{\lambda}\}\} \quad (6)$$
$$\forall x \in X. \exists \alpha \in \boldsymbol{\lambda}. \mathbf{f}\alpha = x \quad (7)$$

Note that (3), (4) and (7) imply $\mathbf{f}$ is a bijection taking $\boldsymbol{\lambda}$ onto X, which will complete the proof.

We begin by proving (1). In the Egal proof we express (1) as a claim and then using `{` to open a subproof.

```
claim Lfeq: forall alpha, f alpha = F alpha f.
```

The subproof proceeds by making and proving the following subclaims.

$$\forall \alpha. \forall hk : \iota \to \iota. (\forall \beta \in \alpha. h\beta = k\beta) \Rightarrow \forall x. \mathbf{P}\alpha x h \Rightarrow \mathbf{P}\alpha x k \quad (8)$$
$$\forall \alpha. \forall hk : \iota \to \iota. (\forall \beta \in \alpha. h\beta = k\beta) \Rightarrow \forall x. \mathbf{P}\alpha x k \Rightarrow \mathbf{P}\alpha x h \quad (9)$$
$$\forall \alpha. \forall hk : \iota \to \iota. (\forall \beta \in \alpha. h\beta = k\beta) \Rightarrow \forall x. \mathbf{Q}\alpha h x \Rightarrow \mathbf{Q}\alpha k x \quad (10)$$
$$\forall \alpha. \forall hk : \iota \to \iota. (\forall \beta \in \alpha. h\beta = k\beta) \Rightarrow \mathbf{Q}\alpha h = \mathbf{Q}\alpha k \quad (11)$$
$$\forall \alpha. \forall hk : \iota \to \iota. (\forall \beta \in \alpha. h\beta = k\beta) \Rightarrow \mathbf{F}\alpha h = \mathbf{F}\alpha k. \quad (12)$$

To prove (8) let α, h, k and x such that $\forall \beta \in \alpha. h\beta = k\beta$ and $\mathbf{P}\ \alpha\ x\ h$ be given. By the definition of $\mathbf{P}$ we know $x \in X$ and $\forall \beta \in \alpha. h\beta \neq x$. Using $\forall \beta \in \alpha. h\beta = k\beta$ we conclude $\forall \beta \in \alpha. k\beta \neq x$ as well and so $\mathbf{P}\ \alpha\ x\ k$. By interchanging the role of h and k we can infer (9) from (8). To prove (10) let α, h, k and x such that $\forall \beta \in \alpha. h\beta = k\beta$ and $\mathbf{Q}\ \alpha\ h\ x$ be given. Then we know $\mathbf{P}\ \alpha\ x\ h$ and $\forall y. \mathbf{P}\ \alpha\ y\ h \Rightarrow \mathbf{V}_x \subseteq \mathbf{V}_y$. We know $\mathbf{P}\ \alpha\ x\ k$ by (8). It remains to prove $\forall y. \mathbf{P}\ \alpha\ y\ k \Rightarrow \mathbf{V}_x \subseteq \mathbf{V}_y$. Let y such that $\mathbf{P}\ \alpha\ y\ k$ be given. By (9) we know $\mathbf{P}\ \alpha\ y\ h$ and so $\mathbf{V}_x \subseteq \mathbf{V}_y$ as desired. We can infer (11) from (10) using extensionality principles (of the higher-order logic, not the set theory). Once we have (11) we can infer (12) by rewriting underneath the choice operator defining $\mathbf{F}$. Finally, (1) follows from (12) and Proposition 5. In the Egal proof, we use `}` to close the subproof of the claim `Lfeq`.

We next proceed with proving (2), (3), (4), (5), (6) and (7). In each case in Egal a corresponding `claim` statement is made and a subproof of the claim is given. In some cases the subproof involves subclaims (described below).

We prove (2) by $\in$-induction. Let α be given and assume as inductive hypothesis $\forall \gamma. \gamma \in \alpha \Rightarrow \gamma \in \boldsymbol{\lambda} \Rightarrow \mathbf{Q}\ \gamma\ \mathbf{f}\ (\mathbf{f}\gamma)$. Assume $\alpha \in \boldsymbol{\lambda}$, i.e., $\alpha \in U$ and $\mathsf{ordinal}\ \alpha$. Under these assumptions we prove the following subclaims:

$$\forall \beta \in \alpha. \mathbf{Q}\ \beta\ \mathbf{f}\ (\mathbf{f}\beta) \ (13)$$
$$\forall \beta \in \alpha. \mathbf{f}\beta \in X \ (14)$$
$$\{\mathbf{f}\beta | \beta \in \alpha\} \subseteq X \ (15)$$
$$\{\mathbf{f}\beta | \beta \in \alpha\} \in U \ (16)$$
$$\exists x. \mathbf{P}\ \alpha\ x\ \mathbf{f} \ (17)$$
$$\exists x. \mathbf{Q}\ \alpha\ \mathbf{f}\ x \ (18)$$
$$\mathbf{Q}\ \alpha\ \mathbf{f}\ (\mathbf{F}\ \alpha\ \mathbf{f}) \ (19)$$

The first claim (13) follows immediately from the inductive hypothesis for $\beta \in \alpha$ using Proposition 3 and transitivity of U. The next claim (14) follows from (13) and the definitions of $\mathbf{Q}$ and $\mathbf{P}$. From (14) we know $f\beta \in X \subseteq U$ for each $\beta \in \alpha$. Hence (15) and (16) follow, using closure of U under replacement for (16).

We prove (17) by contradiction. Assume $\neg\exists x.\mathbf{P}\ \alpha\ x\ \mathbf{f}$. Under this assumption we can prove $\{\mathbf{f}\beta|\beta \in \alpha\} = X$ which contradicts (16) and the assumption that $X \notin U$. By (15) it suffices to prove $X \subseteq \{\mathbf{f}\beta|\beta \in \alpha\}$. Let $x \in X$ be given. Assume $x \notin \{\mathbf{f}\beta|\beta \in \alpha\}$. This assumption can be reformulated as $\forall\beta \in \alpha.\mathbf{f}\beta \neq x$. Combining this with $x \in X$ we have $\mathbf{P}\ \alpha\ x\ \mathbf{f}$, contradicting the assumption that no such x exists. We conclude (17).

We next prove (18). Let $\mathbf{Y}$ be $\{\mathbf{V}_x|x \in X \text{ such that } \forall\beta \in \alpha.\mathbf{f}\beta \neq x\}$. By (17) there is a w such that $\mathbf{P}\ \alpha\ w\ f$. That is, $w \in X$ and $\forall\beta \in \alpha.\mathbf{f}\beta \neq w$. Clearly $\mathbf{V}_w \in \mathbf{Y}$. By Regularity (Proposition 2) there is some $Z \in \mathbf{Y}$ such that $\neg\exists z \in \mathbf{Y}.z \in Z$. Since $Z \in \mathbf{Y}$ there must be some $x \in X$ such that $Z = \mathbf{V}_x$ and $\forall\beta \in \alpha.\mathbf{f}\beta \neq x$. We will prove $\mathbf{Q}\ \alpha\ \mathbf{f}\ x$ for this x. We know $\mathbf{P}\ \alpha\ x\ \mathbf{f}$ since $x \in X$ and $\forall\beta \in \alpha.\mathbf{f}\beta \neq x$. It remains only to prove $\forall y.\mathbf{P}\ \alpha\ y\ \mathbf{f} \Rightarrow \mathbf{V}_x \subseteq \mathbf{V}_y$. Let y such that $\mathbf{P}\ \alpha\ y\ \mathbf{f}$ be given. By Theorem 2:7 either $\mathbf{V}_y \in \mathbf{V}_x$ or $\mathbf{V}_x \subseteq \mathbf{V}_y$. It suffices to prove $\mathbf{V}_y \in \mathbf{V}_x$ yields a contradiction. We know $\mathbf{V}_y \in \mathbf{Y}$ since $\mathbf{P}\ \alpha\ y\ \mathbf{f}$. If $\mathbf{V}_y \in \mathbf{V}_x$, then $\mathbf{V}_y \in Z$ (since $Z = \mathbf{V}_x$), contradicting $\neg\exists z \in \mathbf{Y}.z \in Z$.

We conclude (19) by (18) and the property of the choice operator used in the definition of $\mathbf{F}$. By (19) and (1) we have $\mathbf{Q}\ \alpha\ \mathbf{f}\ (\mathbf{f}\alpha)$. Recall that this was proven under an inductive hypothesis for α. We now discharge this inductive hypothesis and conclude (2).

We can now easily infer (3) from (2) and the definitions of $\mathbf{Q}$ and $\mathbf{P}$.

We next prove (4). Let $\alpha, \beta \in \boldsymbol{\lambda}$ such that $\mathbf{f}\alpha = \mathbf{f}\beta$ be given. By Proposition 4 either $\alpha \in \beta$, $\alpha = \beta$ or $\beta \in \alpha$. To infer $\alpha = \beta$ it suffices to prove a contradiction in the other two cases. By (2) we know $\mathbf{Q}\ \alpha\ \mathbf{f}\ (\mathbf{f}\alpha)$ and $\mathbf{Q}\ \beta\ \mathbf{f}\ (\mathbf{f}\beta)$. Hence $\mathbf{P}\ \alpha\ (\mathbf{f}\alpha)\ \mathbf{f}$ and $\mathbf{P}\ \beta\ (\mathbf{f}\beta)\ \mathbf{f}$. If $\alpha \in \beta$, then $\mathbf{P}\ \beta\ (\mathbf{f}\beta)\ \mathbf{f}$ contradicts $\mathbf{f}\alpha = \mathbf{f}\beta$. If $\beta \in \alpha$, then $\mathbf{P}\ \alpha\ (\mathbf{f}\alpha)\ \mathbf{f}$ contradicts $\mathbf{f}\alpha = \mathbf{f}\beta$.

By (4) we know $\mathsf{bij}\ \boldsymbol{\lambda}\ \{\mathbf{f}\ \alpha|\alpha \in \boldsymbol{\lambda}\}\ \mathbf{f}$. Using Theorem 4 we know (5). From (5) and set extensionality it is easy to prove (6).

Finally we prove (7). Let $x \in X$ be given. Assume $\neg\exists\alpha \in \boldsymbol{\lambda}.\mathbf{f}\alpha = x$. We will prove $\boldsymbol{\lambda} \in \boldsymbol{\lambda}$, contradicting Proposition 1. It is easy to prove $\boldsymbol{\lambda}$ is an ordinal, so it suffices to prove $\boldsymbol{\lambda} \in U$. This is proven via the following subgoals:

$$\forall\alpha \in \boldsymbol{\lambda}.\mathbf{P}\alpha x\mathbf{f}\ \ (20) \qquad \forall\alpha \in \boldsymbol{\lambda}.\mathbf{V}_{\mathbf{f}\alpha} \subseteq \mathbf{V}_x\ \ (21) \qquad \{\mathbf{f}\alpha|\alpha \in \boldsymbol{\lambda}\} \in U\ \ (22)$$

Since $x \in X$ to prove (20) it is enough to argue $\forall\alpha \in \boldsymbol{\lambda}.\forall\beta \in \alpha.f\beta \neq x$. If $\alpha \in \boldsymbol{\lambda}$, $\beta \in \alpha$ and $f\beta = x$, then $\beta \in \boldsymbol{\lambda}$ (by Proposition 3) contradicting our assumption that $\neg\exists\alpha \in \boldsymbol{\lambda}.\mathbf{f}\alpha = x$. To prove (21) let $\alpha \in \boldsymbol{\lambda}$ be given. By (2) we know $\mathbf{Q}\ \alpha\ \mathbf{f}\ (\mathbf{f}\alpha)$ and so $\forall y.\mathbf{P}\ \alpha\ y\ \mathbf{f} \rightarrow \mathbf{V}_{\mathbf{f}\alpha} \subseteq \mathbf{V}_y$. Applying this with x and (20) we have (21). Since $x \in X \subseteq U$ we know $\mathbf{V}_x \in U$ by Theorem 3. Hence $\wp(\wp(\mathbf{V}_x)) \in U$ since U is $\wp$-closed. For each $\alpha \in \boldsymbol{\lambda}$, $\mathbf{f}\alpha \subseteq \mathbf{V}_{\mathbf{f}\alpha}$ by Theorem 2:3 and so $\mathbf{f}\alpha \subseteq \mathbf{V}_x$ by (21). Hence $\{\mathbf{f}\alpha|\alpha \in \boldsymbol{\lambda}\} \in \wp(\wp(\mathbf{V}_x))$. Since U is transitive we conclude (22). Since U is closed under replacement we know $\boldsymbol{\lambda} \in U$ by (22), (5) and (6).

We can now easily conclude Tarski's Axiom A in Egal.

Theorem 5 (Tarski A). *For each set N there exists an M such that*

1. $N \in M$,
2. $\forall X \in M. \forall Y \subseteq X. Y \in M$,
3. $\forall X \in M. \exists Z \in M. \forall Y \subseteq X. Y \in Z$ *and*
4. $\forall X \subseteq M. (\exists f : \iota \to \iota. \mathsf{bij}\ X\ M\ f) \vee X \in M$.

Proof. We use $U := \mathcal{U}_N$ as the witness for M. We know $N \in \mathcal{U}_N$, $\mathcal{U}_N$ is transitive and ZF-closed by the axioms of our set theory. All the properties except the last follow easily from these facts. We focus on the last property. Let $X \subseteq U$ be given. Since we are in a classical setting it is enough to assume $X \notin U$ and prove there is some bijection $f : \iota \to \iota$ taking X onto U. Since $U \subseteq U$ and $U \notin U$ (using Proposition 1), we know there is a bijection g taking $\{\alpha \in U | \mathsf{ordinal}\ \alpha\}$ onto U by Lemma 1. Since $X \subseteq U$ and $X \notin U$, we know there is a bijection h taking $\{\alpha \in U | \mathsf{ordinal}\ \alpha\}$ onto X by Lemma 1. By Theorem 4 there is a bijection g^{-1} taking X onto $\{\alpha \in U | \mathsf{ordinal}\ \alpha\}$. The composition of g^{-1} and h yields a bijection f taking X onto U as desired.

6 Grothendieck Universes in Mizar

In this section we construct Grothendieck universes using notions introduced in the MML articles `CLASSES1` and `CLASSES2` [1,19] primarily by Bancerek. For this purpose, first, we briefly introduce the relevant constructions done by Bancerek. We then define the notion of a Grothendieck universe of a set A as a Mizar type, the type of all transitive sets with A as a member that are closed under power sets and internal family unions. Since Mizar types must be nonempty, we are required to construct such a universe. We finally introduce a functor GrothendieckUniverse A that returns the least set of the type. Additionally, we show that every such Grothendieck universe is closed under replacement formulating the property as a Mizar scheme.

To simplify notation we present selected Mizar operators in more natural ways closer to informal mathematical practice. In particular, we use $\emptyset$, $\in$, $\subseteq$, $\wp$, $|\cdot|$, $\bigcup$ to represent Mizar symbols as {}, in, c=, bool, card, union, respectively.

Following Bancerek, we will start with the construction of the least Tarski universe that contains a given set A. Tarski's Axiom A directly implies that there exists a Tarski set T_A that contains A where Tarski is a Mizar *attribute* (for more details see [13]) defined as follows:

```
attr T is Tarski means :: CLASSES1:def 2
  T is subset-closed & (for X holds X ∈ T implies ℘(X)∈ T) &
  for X holds X ⊆ T implies X,T are_equipotent or X ∈ T;
```

Informally we say that T is Tarski to mean T is closed under subset, power and each subset of T is a member of T or is equipotent with T. Then one shows that $\bigcap\{X | A \in X \subseteq T_A, X \text{ is Tarski set}\}$ is the least (with respect to the inclusion) Tarski set that contains A, denoted by Tarski-Class A.

By definition it is easy to prove the following:

Theorem 6. *The following facts hold.*

1. $\forall A.\ A \in \mathsf{Tarski\text{-}Class}\,A$,
2. $\forall A\,X\,Y.\ Y \subseteq X \wedge X \in \mathsf{Tarski\text{-}Class}\,A \Rightarrow Y \in \mathsf{Tarski\text{-}Class}\,A$,
3. $\forall A\,X.\ Y \in \mathsf{Tarski\text{-}Class}\ A \Rightarrow \wp(X) \in \mathsf{Tarski\text{-}Class}\ A$,
4. $\forall A\,X.\ X \subseteq \mathsf{Tarski\text{-}Class}\,A \wedge |X| < |\mathsf{Tarski\text{-}Class}\,A| \Rightarrow X \in \mathsf{Tarski\text{-}Class}\,A$.

Tarski universes, as opposed to Grothendieck universes, might not be transitive (called epsilon-transitive in the MML) but via transfinite induction Bancerek proved that Tarski-Class A is transitive if A is transitive (cf. Theorems 22 and 23 in [1]). Therefore, in our construction we take the transitive closure of A prior to the application of the Tarski-Class functor. Using a recursion scheme we know for a given set A there exists a recursive sequence f such that $f(0) = A$ and $\forall k \in \mathbb{N}.\ f(k+1) = \bigcup f(k)$. For such an f, $\bigcup\{f(n)|n \in \mathbb{N}\}$ is the least (with respect to the inclusion) transitive set that includes A (or contains A if we start with $f(0) = \{A\}$). The operator is defined in [1] as follows:

```
func the_transitive-closure_of A → set means :: CLASSES1:def 7
  for x holds x ∈ it iff ex f being Function, n being Nat st
    x ∈ f.n & dom f = ℕ & f.0 = A & for k being Nat holds f.(k+1) = ⋃ f.k;
```

We now turn to a formulation of ZF-closed property in Mizar. It is obvious that $\wp$-closed, $\bigcup$-closed properties can we expressed as as two Mizar types as follows:

```
attr X is power-closed means for A being set st A ∈ X holds ℘(A) ∈ X;
attr X is union-closed means for A being set st A ∈ X holds ⋃(A) ∈ X;
```

Note that we cannot express the closure under replacement as a Mizar type since each condition that occurs after **means** has to be a first-order statement. We must therefore use an alternative approach that uses closure under internal family unions using the notion of a function but also its domain (dom) and range (rng) as follows:

```
attr X is FamUnion-closed means
  for A being set for f being Function st dom f = A & rng f ⊆ X & A ∈ X
    holds ⋃ rng f ∈ X;
```

Comparing the properties of Tarski and Grothendieck universes we can prove the following:

Theorem 7. *The following facts hold.*

1. $\forall X.X$ *is* Tarski $\Rightarrow X$ *is* subset-closed power-closed,
2. $\forall X.X$ *is* epsilon-transitive Tarski $\Rightarrow X$ *is* union-closed,
3. $\forall X.X$ *is* epsilon-transitive Tarski $\Rightarrow X$ *is* FamUnion-closed.

Proof. Part 1 is an easy consequences of the Tarski definition and properties of powersets. Part 2 ia a direct conclusion of the MML theorem CLASSES2:59 proven by Bancerek. To prove 3 let X be epsilon-transitive Tarski set, A be set and f be function such that dom $f = A$, rng $f \subseteq X$, $A \in X$. Since X is subset-closed as a Tarski set and $A \in X$, we know that $\wp(A) \subseteq X$. By Cantor's theorem we conclude that $|A| < |\wp(A)|$ and consequently $|A| < |X|$. Since $|\mathsf{rng}\ f| \leq |\mathsf{dom}\ f| = |A|$, we know that $\mathsf{rng}\ f$ is not equipotent with X. Then $\mathsf{rng}\ f \in X$ since X is Tarski and $\mathsf{rng}\ f \subseteq X$, and finally $\bigcup \mathsf{rng}\ f \in X$ by Part 2.

We can now easy infer from Theorem 7 that the term:

$$\mathsf{Tarski\text{-}Class(the\_transitive\text{-}closure\_of}\ \{\mathsf{A}\}) \tag{23}$$

is suitable to prove that the following Mizar type is inhabited:

```
mode Grothendieck of A → set means
  A ∈ it & it is epsilon-transitive power-closed FamUnion-closed;
```

Now it is a simple matter to construct the Grothendieck universe of a given set A ($\mathsf{GrothendieckUniverse}\ A$) since $\bigcap\{X | X \subseteq G_A, X \text{ is } \mathsf{Grothendieck}\ \textbf{of}\ A\}$ is the least (with respect to the inclusion) $\mathsf{Grothendieck}$ **of** A, where G_A denotes the term (23).

As we noted earlier, we cannot express the closure under replacement property as a Mizar type or even assumption in a Mizar theorem. However we can express and prove that every $\mathsf{Grothendieck}$ **of** A satisfies this property as follows:

```
scheme ClosedUnderReplacement
    {A() → set, U() → Grothendieck of A(),F(set) → set}:
  {F(x) where x is Element of A(): x ∈ A()} ∈ U()
provided
  for X being set st X ∈ A() holds F(X) ∈ U()
```

The proof uses a function that maps each x in $\mathsf{A}()$ to $\{\mathsf{F}(x)\}$.

7 Conclusion

We have presented the foundational work required in order to port formalizations from Mizar to Egal or Egal to Mizar. In Egal this required a nontrivial proof of Tarski's Axiom A, an axiom in Mizar. In Mizar this required finding equivalent first-order representations for the relevant higher-order terms and propositions used in Egal and then constructing a Grothendieck universe operator in Mizar.

Acknowledgment

This work has been supported by the European Research Council (ERC) Consolidator grant nr. 649043 *AI4REASON* and the Polish National Science Center granted by decision n°DEC-2015/19/D/ST6/01473.

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